ISRAEL JOURNAL OF MATHEMATICS, Vol. 49, No. 4, 1984

ON THE UNIFORM APPROXIMATION PROPERTY IN BANACH SPACES

BY

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ABSTRACT

We derive a criterion for a Banach space to fail the uniform approximation property (UAP). This criterion is applied to prove that c_p spaces of matrices fail UAP if $p > 80$.

0. A Banach space X has the uniform approximation property (abbreviated UAP) if for every integer k there exists an integer $m(k)$ such that for every k-dimensional subspace E of X there exists an operator $T : X \to X$ such that

$$\|T\| \leqq \lambda, \quad \operatorname{rk} T \leqq m(k) \quad \text{and} \quad T_{|E} = \mathrm{Id}_E$$

for some $\lambda < \infty$ which is independent of k. ($T_{|E}$ denotes the restriction of T to E and Id_E denotes the identity on E.)

This notion was introduced in [4] and has been discussed in some detail in [1], [3], [5]. In [4] it was proved that the spaces L_p have UAP. This result was extended in [3], where it was proved that the reflexive Orlicz spaces have UAP. In [5] it was shown that the spaces of type $(L_p \oplus L_p \oplus \cdots)_{l_q}$ have UAP. In [1] several abstract results were obtained (e.g., that the dual of a space with UAP also has UAP). [1] also generalized the results of [5]. Probably the deepest result concerning UAP is that of P. Jones [2], who proved recently that the space H_1 does have UAP.

Obviously, UAP is a stronger property than the approximation property. Unlike the approximation property, it is usually quite difficult to verify that a given Banach space has UAP (as the results quoted above indicate).

In the present paper we establish some negative criteria for UAP. In sections 1 and 2 we derive a rather convenient criterion for a Banach space not to have

Received November 28, 1983 and in revised form June 21, 1984

UAP. This criterion is applied to the spaces c_p for $p > 80$ (and their dual spaces). We obtain this criterion by an analysis of our paper [6] and in several places we borrow on its methods.

1. Let X be a Banach space; by $F = F(X)$ we denote the space of all finite rank linear operators from X into X. For $T \in F$ we denote by $\| T \|_\vee$ its operator norm and we define

$$\| T \|_{n\wedge} = \inf\{\Sigma \| T_m \|_\vee : T = \Sigma T_m, \operatorname{rk} T_m \leqq n\}.$$

By $X^* \otimes X$ we denote the algebraic tensor product of X^* and X; every $\xi \in X^* \otimes X$ acts on F in the natural way, i.e., for $\xi = \Sigma x_n^* \otimes x_n$ we define

$$\langle \xi, T \rangle = \Sigma x_n^* T x_n \qquad \text{for } T \in F.$$

Given any norm $\| \quad \|$ on F, by $\| \quad \|^*$ we denote its dual norm on $X^* \otimes X$, i.e.,

$$\| \xi \|^* = \sup\{\langle \xi, T \rangle : \| T \| \leqq 1\}.$$

In this way the norms $\| \quad \|_{n\wedge}$ give rise to the norms $\| \quad \|_{n\wedge}^*$ on $X^* \otimes X$.

For $\bar{x} = (x_1, \dots, x_k) \in X^k$ we denote

$$J(\bar{x}) = \{T \in F : Tx_i = x_i \text{ for } i = 1, \dots, k\}$$

and, for $n = 1, 2, \dots$ and $\lambda \geqq 1$, we denote

$$J(\bar{x}, n, \lambda) = \{T \in J(\bar{x}) : \| T \| \leqq \lambda \text{ and } \operatorname{rk} T \leqq n\}.$$

The following lemma is immediate.

LEMMA 1. *Let $\bar{x} = (x_1, \dots, x_k) \in X^k$. If there exists $\xi \in X^* \otimes X$ such that $\| \xi \|_{n\wedge}^* < 1/\lambda$ and*

(1) $$\langle \xi, T \rangle = 1 \qquad \textit{for every } T \in J(\bar{x}),$$

then $J(\bar{x}, n, \lambda) = \emptyset$.

Let us now define a norm $\| \quad \|_n$ on F by

$$\| T \|_n = \max\left\{\frac{1}{n} \| T \|_\wedge, \| T \|_\vee\right\}$$

(here $\| T \|_\wedge = \| T \|_{1\wedge}$).

It is clear that

$$\| T \|_n \leqq \| T \|_{n\wedge} \qquad \text{for every } T \in F$$

(because, if $T = \Sigma T_m$, then $\| T \|_\vee \leqq \Sigma \| T_m \|_\vee$ and, if $\operatorname{rk} T_m \leqq n$, then $\| T_m \|_\wedge \leqq n \| T_m \|_\vee$, thus $\| T \|_\wedge \leqq \Sigma \| T_m \|_\wedge \leqq n \Sigma \| T_m \|_\vee$).

Consequently, we have

$$\| \quad \|_n^* \geqq \| \quad \|_{n\wedge}^*. \tag{2}$$

The norm $\| \quad \|_n^*$ is easy to handle since, obviously,

$$\| \xi \|_n^* \leqq \inf\{ n \| \beta \|_\vee + \| \psi \|_\wedge : \xi = \beta + \psi \}. \tag{3}$$

Let us finally observe that (1) is satisfied if

$$\xi = \sum_{i=1}^k x_i^* \otimes x_i \qquad \text{with } \operatorname{tr} \xi \overset{\text{def}}{=} \Sigma x_i^*(x_i) = 1.$$

This, together with Lemma 1, (2) and (3) gives us the following useful criterion for proving that a given finite dimensional subspace is "badly complemented" and, more generally, that a Banach space does not have UAP.

LEMMA 2. *Let X be a Banach space.*

(i) *If $x_1, \ldots, x_k \in X$, $x_1^*, \ldots, x_k^* \in X^*$ and $\beta, \psi \in X^* \otimes X$ satisfy*

$$\Sigma x_i^*(x_i) = 1, \quad \Sigma x_i^* \otimes x_i = \beta + \psi, \quad \| \beta \|_\vee < 1/2n\lambda, \quad \| \psi \|_\wedge < 1/2\lambda, \tag{4}$$

then $J(\bar{x}, n, \lambda) = \varnothing$.

(ii) *If for every $\eta > 0$ there exists a number $k = k(\eta)$ such that for every $\varepsilon > 0$ there are ξ, β, $\psi \in X^* \otimes X$ satisfying*

$$\operatorname{rk} \xi \leqq k, \quad \operatorname{tr} \xi = 1, \quad \xi = \beta + \psi, \quad \| \beta \|_\vee \leqq \varepsilon, \quad \| \psi \|_\wedge < \eta, \tag{5}$$

then X does not have UAP.

(For $\xi \in X^* \otimes X$ we denote $\operatorname{rk} \xi = \inf\{ k : \xi = \Sigma_{i=1}^k x_i^* \otimes x_i \}$.)

When estimating $\| \quad \|_\vee$ we shall use the following simple inequality:

$$\| \Sigma x_i \otimes y_i \|_\vee \leqq \max_{\varepsilon_i = \pm 1} \| \Sigma \varepsilon_i x_i \| \max_i \| y_i \|. \tag{6}$$

2. Now we shall apply the criterion of Section 1 to a class of Banach spaces.

Let M_n denote the set of all $2^n \times 2^n$ matrices, whose entries are $0, 1, -1$; for $v \in M_n$ its entries will denoted by $v(\varepsilon, \eta)$ where $\varepsilon, \eta \in \{-1, 1\}^n$, i.e., $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $\eta = (\eta_1, \ldots, \eta_n)$ with $\varepsilon_i, \eta_j = \pm 1$. For $v \in M_n$, $w \in M_m$ we define $v \cdot w \in M_{n+m}$ by

$$\begin{aligned} &(v \cdot w)(\varepsilon_1, \ldots, \varepsilon_{n+m}; \eta_1, \ldots, \eta_{n+m}) \\ &\quad = v(\varepsilon_1, \ldots, \varepsilon_n; \eta_1, \ldots, \eta_n) w(\varepsilon_{n+1}, \ldots, \varepsilon_{n+m}; \eta_{n+1}, \ldots, \eta_{n+m}). \end{aligned}$$

We shall write $v \sim w$ if v can be obtained from w by permutations of rows and columns, by multiplication of rows and columns by ± 1 and by deleting (adding) rows and columns, consisting entirely of zeros.

Let us observe that

$$\text{if } v \sim v' \text{ and } w \sim w' \quad \text{then } v \cdot w \sim v' \cdot w'. \tag{7}$$

We define $\omega \in M_1$ by

$$\omega = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and for $n = 1, 2, \ldots$ we define $\omega_n \in M_n$ by

$$\omega_n = \omega \cdot \cdots \cdot \omega \qquad (n \text{ times}),$$

i.e. $\omega_n(\varepsilon_1, \ldots, \varepsilon_n; \eta_1, \ldots, \eta_n) = \omega(\varepsilon_1, \eta_1) \cdot \cdots \cdot \omega(\varepsilon_n, \eta_n)$.

For $\varepsilon, \eta = \{-1, 1\}^n$ define $e_{\varepsilon,\eta} \in M_n$ by $e_{\varepsilon,\eta}(\varepsilon', \eta') = 1$ if $\varepsilon' = \varepsilon$, $\eta' = \eta$, $e_{\varepsilon,\eta}(\varepsilon', \eta') = 0$ otherwise.

Let us denote

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \vartheta = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau_n = \tau \cdot \cdots \cdot \tau, \quad \vartheta_n = \vartheta \cdot \cdots \cdot \vartheta \quad (n \text{ times})$$

and let $\chi_k \in M_k$ be the matrix, all of whose entries are 1 (thus $\chi_k \sim \tau_k \cdot \vartheta_k$).

For $A \subset \{-1, 1\}^n$ let us define $\omega_n^A, \omega_A^n \in M_n$ by

$$\omega_A^n(\varepsilon, \eta) = \begin{cases} \omega_n(\varepsilon, \eta) & \text{if } \eta \in A, \\ 0 & \text{if } \eta \notin A, \end{cases}$$
$$\omega_n^A(\varepsilon, \eta) = \begin{cases} \omega_n(\varepsilon, \eta) & \text{if } \varepsilon \in A, \\ 0 & \text{if } \varepsilon \notin A. \end{cases} \tag{8}$$

Let us denote $M^\infty = \bigcup_{n=1}^\infty M_n$. A function on M^∞ whose restriction to each M_n is a norm, will be called a norm on M^∞. Let $\| \ \|$ be a norm on M^∞. $\| \ \|$ will be called *admissible* if

$$\begin{aligned} \| \tau_k \cdot \vartheta_m \cdot \omega_n \| &= \| \vartheta_m \cdot \omega_n \cdot \tau_k \| = \| \omega_n \cdot \tau_k \cdot \vartheta_m \| \\ &= \| \vartheta_m \cdot \tau_k \cdot \omega_n \| = \| \tau_k \cdot \omega_n \cdot \vartheta_m \| = \| \omega_n \cdot \vartheta_m \cdot \tau_k \| \\ &= \| \tau \|^k \| \vartheta \|^m \| \omega_n \| \qquad \text{for } n, k, m = 0, 1, 2, \ldots, \end{aligned} \tag{9}$$

$$v \sim w \quad \text{implies} \quad \| v \| = \| w \|, \tag{10}$$

(11) $\quad \|\omega_A^n\|$ depends on $|A|$ only $\quad$ and $\quad \|\omega_n^A\|$ depends on $|A|$ only

(by $|A|$ we denote the cardinality of A).

Let X be a Banach space. Suppose that there is a mapping $v \rightsquigarrow v^0$ from M^∞ into X and another mapping $v \rightsquigarrow v^*$ from M^∞ into X^*. We define the norms $\| \;\|^0$ and $\| \;\|^*$ on M^∞ by

$$\|v\|^0 = \|v^0\|, \qquad \|v\|^* = \|v^*\|.$$

Proposition. *With the above notation, suppose that* $\| \;\|^0$ *and* $\| \;\|^*$ *are admissible norms on* M^∞. *Let, moreover,*

(12) $$e^*_{\xi,\eta}(e^0_{\xi',\eta'}) = 1 \quad \textit{if } \varepsilon = \varepsilon',\ \eta = \eta'; \qquad e^*_{\xi,\eta}(e^0_{\xi',\eta'}) = 0 \quad \textit{otherwise},$$

$$\|e_{\xi,\eta}\|^0 = \|e_{\xi,\eta}\|^* = 1 \qquad \textit{for all } \varepsilon,\ \eta;$$

(13) $$\|\tau\|^0\|\tau\|^* = \|\vartheta\|^0\|\vartheta\|^* = 2, \quad \|\omega_n\|^0\|\omega_n\|^* = 4^n,$$

(14) $$\sup_n \frac{\|\omega_{n+1}\|^0}{\|\omega_n\|^0} < \min(\|\tau\|^0, \|\vartheta\|^0)^{41/40}.$$

Then X does not have UAP.

Application. Let X be a space of compact operators on the Hilbert space l_2. We can index the unit vector basis of l_2 by the countable set $\bigcup_{n=1}^\infty \{-1,1\}^n$ and embed M_n into X by $e^0_{\xi,\eta}$ = the operator which maps the ε-th unit vector of l_2 into the η-th unit vector and the others into 0. Also $e^*_{\xi,\eta}$ is defined naturally by

$$e^*_{\xi,\eta}\langle T\rangle = \langle Te_\varepsilon, e_\eta\rangle.$$

Suppose that the norm of $T \in X$ depends only on the s-numbers of T (the s-numbers of T are the eigenvalues of $|T| = (T^*T)^{1/2}$, listed with their multiplicities, in decreasing order), denoted $(s_i)_{i=1}^\infty$. In other words,

$$\|T\| = |||(s_i)_{i=1}^\infty|||$$

where $||| \;|||$ is a symmetric norm, defined on c_0 ($||| \;|||$ is allowed to take the value ∞).

In this case X is called the *symmetrically normed ideal* (of operators) associated to the symmetric norm $||| \;|||$.

We see that (12) is equivalent to

$$|||(1,0,0,\ldots)||| = 1$$

and that, if this is satisfied, then the conditions (9)–(13) hold. Let us check, for example, the condition (9). The s-numbers of each of the six operators which appear in (9) are

$$(\underbrace{2^{\frac{1}{2}(k+m+n)},\dots,2^{\frac{1}{2}(k+m+n)}}_{2^n \text{ times}},0,\dots),$$

the s-numbers of τ, ν, ω_n are $(2^{1/2},0,0,\dots)$; $(2^{1/2},0,0,\dots)$; $(2^{n/2},\dots,2^{n/2},0,\dots)$, respectively; thus

$$\|\tau\| = \|\nu\| = 2^{1/2}, \quad \|\omega_n\| = 2^{n/2}\|(1,1,\dots,1,0,\dots)\|$$

and so all the entities that appear in (9) are equal to $2^{(k+m+n)/2}\|(1,\dots,1,0,\dots)\|$ where $\|\ \|$ stands either for $|||\ |||$ or for $|||\ |||^*$.

The condition (10) is obviously satisfied, because $v \sim w$ implies the same s-numbers for v and w. The s-numbers of ω_A^n and of ω_n^A are $(2^{n/2},\dots,2^{n/2},0,\dots)$, $|A|$ times, so (11) is also evident. (13) follows from the above computations.

The condition (14) does depend on the norm $|||\ |||$ and is not always satisfied. For example, if $|||\ |||$ is the l_2 norm, then the resulting X is the space of Hilbert–Schmidt operators, which is itself a Hilbert space and thus has the UAP. Let us denote $\lambda(k) = |||(1,\dots,1,0,\dots)|||$, k times. We see easily that (14) is equivalent to

$$(*) \qquad \sup_j \frac{\lambda(2^{j+1})}{\lambda(2^j)} < 2^{1/80}.$$

We have thus obtained as a corollary from our Proposition the following

THEOREM. *Let X be the symmetrically normed ideal, associated to a symmetric norm $|||\ |||$. If*

$$\lambda(1) = 1 \quad \textit{and} \quad \sup_j \frac{\lambda(2^{j+1})}{\lambda(2^j)} < 2^{1/80}$$

then X does not have UAP.

In particular, if X is a c_p-space, i.e. if $|||(x_1,x_2,\dots)||| = (\Sigma |x_j|^p)^{1/p}$, then $(*)$ is equivalent to $p > 80$. Thus

COROLLARY. *The spaces c_p for $p > 80$ (and, dually, for $1 \leqq p < 80/79$) do not have UAP.*

The remainder of the paper is devoted to the proof of the Proposition.

3. Let natural numbers $m < N$ be fixed. Let us define sets $S_a \subset \{-1,1\}^N$ for $a \in \{-1,1\}^m$ by

$$S_a = \{(\varepsilon_1,\dots,\varepsilon_N) \in \{-1,1\}^N : \varepsilon_1 = a_1,\dots,\varepsilon_m = a_m\}.$$

Given $v \in M_N$ we define $v_{a,b} \in M_N$ by

$$v_{a,b}(\varepsilon,\eta)=\begin{cases} v(\varepsilon,\eta) & \text{if } \varepsilon \in S_a, \quad \eta \in S_b \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

and we define $\xi \in X^* \otimes X$ by

$$\xi = 2^{-2N} \sum_{a,b \in \{-1,1\}^m} v^*_{a,b} \otimes v^0_{a,b}.$$

Evidently

$$\operatorname{rk} \xi \leqq 2^{2m}. \tag{16}$$

By (12), we have

(17) if v is unimodular (i.e., all its entries are ± 1), then $\operatorname{tr} \xi = 1$.

Let us define $\beta \in X^* \otimes X$ by

$$\beta = 2^{-2N} \sum_{\varepsilon,\eta \in \{-1,1\}^N} e^*_{\varepsilon,\eta} \otimes e^0_{\varepsilon,\eta}.$$

Let us observe that

$$\|\beta\|_\vee \leqq 2^{-N/2}. \tag{18}$$

Indeed, by (6) we have

$$\|\beta\|_\vee \leqq 2^{-2N} \max_{\varepsilon,\eta} \|e^*_{\varepsilon,\eta}\| \cdot \max_{\alpha_{\varepsilon,\eta}=\pm 1} \|\Sigma \alpha_{\varepsilon,\eta} e^0_{\varepsilon,\eta}\|.$$

The first term equals 1, by (12). The second term is, by the triangle inequality, smaller than

$$2^N \max_\eta \max_{\alpha_\varepsilon = \pm 1} \left\| \sum_\varepsilon \alpha_\varepsilon e^0_{\varepsilon,\eta} \right\|.$$

The norm above is the norm of a matrix which has zeros except for one column, consisting of ± 1. Such a matrix is evidently $\sim$ to ϑ_N and thus

$$\|\beta\|_\vee \leqq 2^{-N} \|\vartheta^0_N\|.$$

Similarly we get $\|\beta\|_\vee \leqq 2^{-N} \|\vartheta^*_N\|$. Hence, by (9) and (13),

$$\|\beta\|_\vee \leqq (2^{-N}\|\vartheta^0_N\| \cdot 2^{-N}\|\vartheta^*_N\|)^{1/2} = 2^{-N/2}.$$

4. Let $n_1, n_2, \dots$ be natural numbers such that

(19) $$n_1 = m, \qquad 32 \mid n_j \text{ for every } j,$$

(20) $$n_{j+1} \leqq 2n_j \qquad \text{for every } j.$$

For economy of space let us denote

$$\omega^j = \omega_{n_j/2}, \qquad D_j = \{-1,1\}^{n_j}, \qquad E_j = \{-1,1\}^{n_j/2}.$$

For $a \in \{-1,1\}^{n_j/4}$ we define a set $S_j^a \subset E_j$ by

(21) $$S_j^a = \{(\varepsilon_1, \dots, \varepsilon_{n_j/2}) \in E_j : \varepsilon_{n_j/4+1} = a_1,\ \varepsilon_{n_j/4+2} = a_2, \dots, \varepsilon_{n_j/2} = a_{n_j/4}\}.$$

We shall construct a unimodular matrix v so that $\|\xi - \beta\|_\wedge$, where ξ and β are defined as above, is small.

Let k be a natural number. A partition Δ of $\{-1,1\}^{2k}$ will be called *regular* if every member of Δ has 2^k elements.

We shall use the following combinatorial fact from [6] (cf. Sublemma 4 there).

LEMMA 3. *Let n be a natural number such that $16 \mid n$. There exist regular partitions ∇_ε for $\varepsilon = 1,2,3,\dots,2^{4n}$ of the set $\{-1,1\}^n$ such that*

$$|A \cap B| \leqq 2^{7n/16} \qquad \textit{for every } A \in \nabla_\varepsilon,\quad B \in \nabla_\eta \textit{ with } \varepsilon \neq \eta.$$

We shall also use the following obvious observation:

(22) if $\mathscr{A}$ and $\mathscr{X}$ are arbitrary regular partitions of $\{-1,1\}^n$, then there exists a permutation ρ of $\{-1,1\}^n$ such that $\rho(A) \in \mathscr{X}$ for every $A \in \mathscr{A}$.

By (20), Lemma 3 and (22), there exist regular partitions ∇_ε^j and permutations ρ_ε^j for $\varepsilon \in D_{j+1}$ of the set E_j such that

(23) $$|A \cap B| \leqq 2^{7n_j/32} \qquad \text{for every } A \in \nabla_\varepsilon^j,\quad B \in \nabla_\eta^j \text{ with } \varepsilon \neq \eta,$$

(24) for every $A \in \nabla_\eta^j$, the set $\rho_\eta^j(A)$ equals one of the sets S_j^a, defined in (21).

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n_j}) \in D_j$ we define $\hat{\varepsilon}, \check{\varepsilon} \in E_j$ by

$$\hat{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{n_j/2}), \qquad \check{\varepsilon} = (\varepsilon_{n_j/2+1}, \dots, \varepsilon_{n_j}).$$

Define now a function $\varphi_j : E_j \times E_j \times D_{j+1} \to \{-1,1\}$ by

$$\varphi_j(\varepsilon', \varepsilon'', h) = \omega^j(\varepsilon', \rho_h^j(\varepsilon'')) \qquad \text{for } \varepsilon', \varepsilon'' \in E_j,\quad h \in D_{j+1},$$

Without loss of generality we can assume that $N = n_1 + n_2 + \cdots + n_q$ for some q.

For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in \{-1,1\}^N$ we denote

$$\varepsilon^j = (\varepsilon_{n_1+\cdots+n_{j-1}+1}, \varepsilon_{n_1+\cdots+n_{j-1}+2}, \ldots, \varepsilon_{n_1+\cdots+n_j}) \in D_j$$

and finally we define

$$v(\varepsilon,\eta) = \prod_{j=1}^{q-1} \varphi_j(\check{\eta}^j, \hat{\varepsilon}^j, \varepsilon^{j+1})\varphi_j(\check{\varepsilon}^j, \hat{\eta}^j, \eta^{j+1}) \cdot \omega_{n_q}(\varepsilon^q, \eta^q). \tag{25}$$

5. In order to estimate the norm $\|\xi - \beta\|_\wedge$ we shall need a few technical lemmas.

We shall write $v \approx w$ if v can be obtained from w by permutation of columns and multiplication of columns by ± 1.

We shall write $u < w$ if $u \approx u'$ where u' can be obtained from w by deleting some columns (i.e., replacing them by zeros).

Let us observe that, by (10) and by the triangle inequality,

$$u < w \quad \text{implies} \quad \|u^0\| \leqq \|w^0\| \quad \text{and} \quad \|u^*\| \leqq \|w^*\|. \tag{26}$$

LEMMA 4. *Let $u \in M_n$, let $z \in M_m$ and let $z_\eta \in M_m$ for $\eta \in \{-1,1\}^n$ be such that $z_\eta < z$. Define $y \in M_{n+m}$ by*

$$y(\varepsilon, \varepsilon', \eta, \eta') = u(\varepsilon,\eta) \cdot z_\eta(\varepsilon', \eta') \quad \textit{for } \varepsilon, \eta \in \{-1,1\}^n, \quad \varepsilon', \eta' \in \{-1,1\}^m. \tag{27}$$

Then $y < u \cdot z$.

PROOF. We just use the definitions: $z_\eta < z$ means that there exist a function $\delta_\eta : \{-1,1\}^m \to \{-1,0,1\}$ and a permutation ρ_η of the set $\{-1,1\}^m$ so that

$$z_\eta(\varepsilon', \eta') = \delta_\eta(\eta') \cdot z(\varepsilon', \rho_\eta(\eta')).$$

Consequently, if we denote

$$\delta(\eta, \eta') = \delta_\eta(\eta'), \quad \rho(\eta, \eta') = (\eta, \rho_\eta(\eta')),$$

we obtain

$$y(\varepsilon, \varepsilon'; \eta, \eta') = \delta(\eta, \eta') \cdot (u \cdot z)(\varepsilon, \varepsilon'; \rho(\eta, \eta')),$$

i.e., $y < u \cdot z$.

Let k be a natural number. We define subsets S_a, S^a of $\{-1,1\}^{2k}$ for $a \in \{-1,1\}^k$ by

$$S_a = \{(\varepsilon_1, \ldots, \varepsilon_{2k}) \in \{-1,1\}^{2k} : \varepsilon_1 = a_1, \ldots, \varepsilon_k = a_k\},$$

$$S^a = \{(\varepsilon_1, \ldots, \varepsilon_{2k}) \in \{-1,1\}^{2k} : \varepsilon_{k+1} = a_1, \ldots, a_{2k} = a_k\}.$$

Let $S, B \subset \{-1,1\}^{2k}$. We define a matrix $\omega_{S,B} \in M_{2k}$ by

$$\omega_{S,B}(\varepsilon,\eta) = \begin{cases} \omega_{2k}(\varepsilon,\eta) & \text{if } \varepsilon \in S, \quad \eta \in B, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5. *Let* $a \in \{-1,1\}^k$.

(i) *For every* $b \in \{-1,1\}^k$ *we have* $\omega_{S_a,S^b} \sim \chi_k$.

(ii) *If* $B \subset \{-1,1\}^{2k}$, *then* $\omega_{S_a,B} < \omega_k \cdot \tau_\alpha$

where α *is a natural number such that*

$$|B \cap S_b| \leqq 2^\alpha \qquad \textit{for every } b \in \{-1,1\}^k.$$

PROOF. (i) We have for $\varepsilon, \eta \in \{-1,1\}^{2k}$

$$\omega_{S_a,S^b}(\varepsilon,\eta) = \omega(a_1,\eta_1)\omega(a_2,\eta_2)\cdot\cdots\cdot\omega(a_k,\eta_k)\omega(\varepsilon_{k+1},b_1)\cdot\cdots\cdot\omega(\varepsilon_{2k},b_k)$$

if $\varepsilon_1 = a_1,\dots,\varepsilon_k = a_k$, $\eta_{k+1} = b_1,\dots,\eta_{2k} = b_k$ and $\omega_{S_a,S^b}(\varepsilon,\eta) = 0$ otherwise. This means that ω_{S_a,S^b} is obtained from χ_k by multiplying its columns by the numbers $\omega(a_1,\eta_1),\dots,\omega(a_k,\eta_k)$ and by multiplying its rows by the numbers $\omega(\varepsilon_{k+1},b_1),\dots,\omega(\varepsilon_{2k},b_k)$, respectively, by adding to it $2^{2k}-2^k$ columns and rows, consisting entirely of zeros and by appropriately permuting (shifting) its columns and rows.

(ii) We have

$$\omega_{S_a,B} = \sum_{b \in \{-1,1\}^k} \omega_{S_a,B\cap S^b}.$$

For $b \in \{-1,1\}^k$ let x_b denote the b-th column of ω_k, i.e.

$$x_b(\xi_1,\dots,\xi_k) = \omega(\xi_1,b_1)\cdot\cdots\cdot\omega(\xi_k,b_k),$$

and let the vector y_b be defined for $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{2k}) \in \{-1,1\}^{2k}$ by

$$y_b(\varepsilon_1,\dots,\varepsilon_{2k}) = \begin{cases} x_b(\varepsilon_{k+1},\dots,\varepsilon_{2k}) & \text{if } \varepsilon_1 = a_1,\dots,\varepsilon_k = a_k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\omega_{S_a,B} \approx u$ where $u \in M_{2k}$ is a matrix each of whose columns is equal to one of the y_b's, $b \in \{-1,1\}^k$ and such that the number of columns equal to y_b is $|B \cap S^b|$. Consequently, $\omega_{S_a,B} < u'$ where u' is a matrix each of whose columns is equal to one of y_b and the number of columns equal to a particular y_b is 2^α. But, clearly, $u' \sim \omega_k \cdot \tau_\alpha$.

Let $l \leqq q$. For $\varepsilon^j, \eta^j \in D_j$, $j = 1,\dots,q$ we set

$$O^l(\varepsilon^l,\dots,\varepsilon^q;\eta^l,\dots,\eta^q)=\prod_{j=l}^{q-1}\varphi_j(\check\eta^j,\hat\varepsilon^j,\varepsilon^{j+1})\varphi_j(\check\varepsilon^j,\hat\eta^j,\eta^{j+1})\cdot\omega_{n_q}(\varepsilon^q,\eta^q)$$

(thus $v = O^1$, cf. (25)).

LEMMA 6. *$O^l \sim \omega_{n_l+\cdots+n_q}$ for $l = 1,\dots,q$.*

PROOF. We use downward induction on l. For $l = q$ the claim is evident. Suppose it is true for some $l \leqq q$. Let $O \in M_{n_{l-1}/2+n_l+\cdots+n_q}$ be defined by

$$O(\varepsilon',\varepsilon^l,\dots,\varepsilon^q;\eta',\eta^l,\dots,\eta^q)=\varphi_{l-1}(\varepsilon',\eta',\eta^l)\cdot O^l(\varepsilon^l,\dots,\varepsilon^q;\eta^l,\dots,\eta^q)$$

for $\varepsilon',\eta'\in E_{l-1}$, $\varepsilon^j,\eta^j\in D_j$, $j=l,\dots,q$. By Lemma 4, $O\sim\omega_{n_l+\cdots+n_q}\cdot\omega^{l-1}$, which is $\sim$ to $\omega_{n_{l-1}/2+n_l+\cdots+n_q}$.

Since we have

$$\begin{aligned}&O^{l-1}(\varepsilon^{l-1},\dots,\varepsilon^q;\eta^{l-1},\dots,\eta^q)\\&\quad=\varphi_{l-1}(\check\eta^{l-1},\hat\varepsilon^{l-1},\varepsilon^l)\cdot O(\check\varepsilon^{l-1},\dots,\varepsilon^q;\hat\eta^{l-1},\dots,\eta_q),\end{aligned}$$

the same argument as above shows that $O^{l-1}\sim\omega_{n_{l-1}+n_l+\cdots+n_q}$.

LEMMA 7. *Let $A\subset\{-1,1\}^n$, $|A|=2^\alpha$. Let the matrices ω_A^n, ω_n^A be defined by* (8). *We have*

$$\|\omega_A^n\|^*=\|\omega_\alpha\|^*\cdot\|\vartheta_{n-\alpha}\|^*,$$

$$\|\omega_n^A\|^*=\|\omega_\alpha\|^*\cdot\|\tau_{n-\alpha}\|^*.$$

PROOF. By (11), $\|\omega_A^n\|^*=\|\omega_B^n\|^*$ where

$$B=\{(\varepsilon_1,\dots,\varepsilon_n)\in\{-1,1\}^n:\varepsilon_{\alpha+1}=\varepsilon_{\alpha+2}=\cdots=\varepsilon_n=1\}.$$

It is easy to see that $\omega_B^n\sim\omega_\alpha\cdot\vartheta_{n-\alpha}$ and this, by (9), implies the first equality. The proof of the second one is analogous.

6. Our main task is to prove that $\|\xi-\beta\|_\wedge$ is small. With this aim we define $\beta_j\in X^*\otimes X$ by

$$\beta_j=2^{-2N}\sum_{a,b\in\{-1,1\}^{N_j}}v^*_{a;b}\otimes v^0_{a;b}$$

where we denote

$$N_j=n_1+n_2+\cdots+n_j,\qquad j=1,\dots,q$$

(let us recall that $N=N_q$) and $v_{a,b}\in M_n$ for $a,b\in\{-1,1\}^m$ are defined by (15).

We have thus $\beta_1 = \xi$, $\beta_q = \beta$.

Let us denote $\zeta = \min(\|\tau^0\|, \|\vartheta^0\|)$ and let λ be such that

$$\sup_n \frac{\|\omega_{n+1}\|}{\|\omega_n\|} = \zeta^{1+\lambda}.$$

We shall prove that

$$\|\beta_j - \beta_{j+1}\| \leqq 2\zeta^{\lambda(n_{j+1}+n_j/4)-n_j/32}. \tag{28}$$

The proof of (28) will take most of the remainder of the paper.

First we shall define a convenient representation of $\beta_j - \beta_{j+1}$. Here the argument is very similar to the corresponding argument in [6].

Let us denote

$$D = \{(a,c) : a \in \{-1,1\}^{N_j}, c \in \{-1,1\}^{N_{j+1}}\},$$

$$D^1 = \{(a,c) : a \in \{-1,1\}^{N_{j+1}}, c \in \{-1,1\}^{N_{j+1}}\}.$$

Let $\gamma = 2^{-2N} \sum_{(a,c)\in D} v^*_{a;c} \otimes v^0_{a;c}$. Let us denote for $a, b \in \{-1,1\}^{N_j}$, $c \in \{-1,1\}^{N_{j+1}}$, $g \in D_{j+1}$:

$$y_{a;b,g} = v^0_{a;b} - v^0_{a;b;g}, \qquad y_{a,g;c} = v^0_{a;c} - v^0_{a,g;c}.$$

We have

$$\beta_j - \gamma = 2^{-2N} \sum_{(a,c)\in D} v^*_{a;c} \otimes y_{a;c}, \qquad \gamma - \beta_{j+1} = 2^{-2N} \sum_{(a,c)\in D^1} v^*_{a;c} \otimes y_{a;c}.$$

Next we define for $\alpha, \beta \in \{-1,1\}^{N_{j-1}}$ and $e, f \in E_j$; $A, S \subset E_j$; $g \in D_{j+1}$:

$$H = H_{g,e,f,A,S,\alpha,\beta} = \{(a,c) \in D : (a^1,\dots,a^{j-1}) = \alpha, (c^1,\dots,c^{j-1}) = \beta, \hat{a}^j = e, \check{c}^j = f,$$
$$\check{a}^j \in S, \hat{c}^j \in A, c^{j+1} = g\},$$

$$H^1 = H^1_{g,e,f,A,S,\alpha,\beta} = \{(a,c) \in D^1 : (c^1,\dots,c^{j-1}) = \alpha, (a^1,\dots,a^{j-1}) = \beta, \hat{c}^j = e, \check{a}^j =$$
$$f, \check{c}^j \in S, \hat{a}^j \in A, a^{j+1} = g\};$$

we recall that, for $a \in \{-1,1\}^{N_i}$, $i = 1,2,\dots,q$, we denote for $j = 1,2,\dots,i$,

$$a^j = (a_{N_{j-1}+1}, a_{N_{j-1}+2}, \dots, a_{N_j-1}, a_{N_j})$$

and

$$\delta = \delta_{g,e,f,A,S,\alpha,\beta} = \sum_{(a,c)\in H} v^*_{a;c} \otimes y_{a;c},$$

$$\delta^1 = \delta^1_{g,e,f,A,S,\alpha,\beta} = \sum_{(a,c)\in H^1} v^*_{a;c} \otimes y_{a;c}.$$

LEMMA 8. *We have*

$$\|\delta\|_\wedge \leqq \Big\| \sum_{(a,c)\in H} v^*_{a:c} \Big\| \Big\| \sum_{(a,c)\in H} y_{a:c} \Big\|,$$

$$\|\delta^1\|_\wedge \leqq \Big\| \sum_{(a,c)\in H^1} v^*_{a:c} \Big\| \Big\| \sum_{(a,c)\in H^1} y_{a:c} \Big\|.$$

PROOF. Let E denote the set of all functions from $\{-1,1\}^{N_j}$ into $\{-1,1\}$ and let F denote the set of all functions from $\{-1,1\}^{N_{j+1}}$ into $\{-1,1\}$.

We have the following identities:

$$\delta = |E|^{-1}|E|^{-1} \sum_{\varepsilon\in E} \sum_{\eta\in E} \Big[\Big(\sum_H \varepsilon(a)\eta(b)v^*_{a:b,g}\Big)\otimes\Big(\sum_H \varepsilon(a)\eta(b)y_{a:b,g}\Big)\Big],$$

$$\delta^1 = |E|^{-1}|F|^{-1} \sum_{\varepsilon\in E} \sum_{\eta\in F} \Big[\Big(\sum_{H^1} \varepsilon(a)\eta(b)v^*_{a,g:b}\Big)\otimes\Big(\sum_{H^1} \varepsilon(a)\eta(b)y_{a,g:b}\Big)\Big]$$

(these formulas are simple applications of the invariance of trace, cf. the proof of Lemma 2 in [6]).

Now let us observe that the matrix $\Sigma_H\,\varepsilon(a)\eta(b)v_{a:b,g}$ is obtained from the matrix $\Sigma_H\, v_{a:b,g}$ by multiplication of rows and columns by ± 1. By (10), this implies

$$\Big\|\sum_H \varepsilon(a)\eta(b)\varphi_{a:b,g}\Big\| = \Big\|\sum_H \varphi_{a:b,g}\Big\|.$$

Similar considerations yield the equalities

$$\Big\|\sum_{H^1} \varepsilon(a)\eta(b)\varphi_{a,g:b}\Big\| = \Big\|\sum_{H^1} \varphi_{a,g:b}\Big\|,$$

etc. This proves Lemma 8.

7. Let $A, S \subset D_j$ be fixed. Let us define $E, F \subset \{-1,1\}^N$ and $F_g, F_g^- \subset \{-1,1\}^N$ for $g \in D_{j+1}$ by

$$E = \{a \in \{-1,1\}^N : (a^1,\ldots,a^{j-1}) = \alpha,\ \hat{a}_j = e,\ \check{a}_j \in S\},$$

$$F = \{b \in \{-1,1\}^N : (b^1,\ldots,b^{j-1}) = \beta,\ \check{b}_j = f,\ \hat{b}_j \in A\},$$

$$F_g = \{b \in \{-1,1\}^N : (b^1,\ldots,b^{j-1}) = \beta,\ \check{b}_j = f,\ \hat{b}_j \in A,\ b^{j+1} = g\},$$

$$F_g^- = F - F_g.$$

For a matrix $u \in M_n$ and for $B, C \subset \{-1, 1\}^N$ we define $u_{B,C} \in M_N$ by

$$u_{B,C}(\varepsilon, \eta) = \begin{cases} u(\varepsilon, \eta) & \text{if } \varepsilon \in B, \quad \eta \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Let us notice that

$$\sum_{(a,c)\in H} v^*_{a;c} = (v_{E,F_g})^*, \qquad \sum_{(a,c)\in H^1} v^*_{a;c} = (v_{F_g,E})^*, \tag{29}$$
$$\sum_{(a,c)\in H} y_{a;c} = (v_{E,F_q^-})^0, \qquad \sum_{(a,c)\in H^1} y_{a,c} = (v_{F_g^-,E})^0.$$

Observe that we have

$$v_{E,F}(\varepsilon, \eta) = \pm\, \varphi_{j-1}(\check{\beta}^{j-1}, \hat{\alpha}^{j-1}, \varepsilon^j)\cdot \varphi_{j-1}(\check{\alpha}^{j-1}, \hat{\beta}^{j-1}, \eta^j)\cdot \varphi_j(f, e, \varepsilon^{j+1})$$
$$\cdot\, \varphi_j(\check{\varepsilon}^j, \hat{\eta}^j, \eta^{j+1})\cdot O^{j+1}(\varepsilon^{j+1}, \dots, \varepsilon^q; \eta^{j+1}, \dots, \eta^q),$$

if $\check{\varepsilon}^j \in S$ and $\hat{\eta}^j \in A$ and $v_{E,F}(\varepsilon, \eta) = 0$ otherwise. Hence,

$$v_{E,F_g} \sim u_{T,A_g} \quad \text{and} \quad v_{E,F_g^-} \sim u_{T,A_g^-},$$

where $u \in M_{n_j/2+n_{j+1}+\dots+n_q}$ and A_g, A_g^-, $T \subset \{-1, 1\}^{n_j/2+n_{j+1}+\dots+n_q}$ are defined by

$$u(\varepsilon', \varepsilon^{j+1}, \dots, \varepsilon^q; \eta', \eta^{j+1}, \dots, \eta^q)$$
$$= \varphi_j(\varepsilon', \eta', \eta^{j+1}) O^{j+1}(\varepsilon^{j+1}, \dots, \varepsilon^q; \eta^{j+1}, \dots, \eta^q),$$
$$A_g = \{(\varepsilon', \varepsilon^{j+1}, \dots, \varepsilon^q) : \varepsilon' \in A, \varepsilon^{j+1} = g\},$$
$$A_g^- = \{(\varepsilon', \varepsilon^{j+1}, \dots, \varepsilon^q) : \varepsilon' \in A, \varepsilon^{j+1} \neq g\},$$
$$T = \{(\varepsilon', \varepsilon^{j+1}, \dots, \varepsilon^q) : \varepsilon' \in S\}.$$

(Here, as usual, ε', η' denote elements of E_m and ε^i, η^i denote elements of D_i.)

LEMMA 9. *Let $S = S_a$ for some $a \in \{-1, 1\}^{n/4}$ and let $A \in \nabla^j_g$. Then*

(i)
$$\| v_{E,F_g} \|^* = \| \omega_{n_{j+1}+\dots+n_q} \|^* \| \vartheta_{n_{j+1}} \|^* \| \chi_{n_j/4} \|^*,$$
$$\| v_{F_g,E} \|^* = \| \omega_{n_{j+2}+\dots+n_q} \|^* \| \tau_{n_{j+1}} \|^* \| \chi_{n_j/4} \|^*;$$

(ii)
$$\| v_{E,F_g} \|^0 \leqq \| \omega_{n_{j+1}+\dots+n_q} \|^0 \| \omega_{n_j/4} \|^0 \| \tau_{7n_j/32} \|^0,$$
$$\| v_{F_g^-,E} \|^0 \leqq \| \omega_{n_{j+1}+\dots+n_q} \|^0 \| \omega_{n_j/4} \|^0 \| \vartheta_{7n_j/32} \|^0.$$

PROOF. (i) By the preceding remarks, we have

$$\| v_{E,F_g} \|^* = \| u_{T,A_g} \|^*.$$

For $h \in D_{j+1}$ define $u_h \in M_{n_j/2}$ by

$$u_h(\varepsilon, \eta) = \omega^j(\varepsilon, \rho_h(\eta)) \qquad (= \varphi_j(\varepsilon, \eta, h)).$$

Evidently,

$$u_{T,A_g} \sim (u_g)_{S,A} \cdot O',$$

where O' is defined for $\varepsilon^i \in D_i$ by

$$O'(\varepsilon^{j+1}, \dots, \varepsilon^q; \eta^{j+1}, \dots, \eta^q) = \begin{cases} O^{j+1}(\varepsilon^{j+1}, \dots, \varepsilon^q; \eta^{j+1}, \dots, \eta^q) & \text{if } \varepsilon^{j+1} = g, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 5(i) and by the fact that $A \in \nabla_g^j$ and thus $\rho_\varepsilon^j(A) \in S_j^a$ for some $a \in \{-1, 1\}^{n/4}$, we get

$$\|(u_g)_{S,A}\|^* = \|\chi_{n_j/4}\|^*.$$

By Lemmas 6 and 7 we obtain the first equality in (i). The second one is completely analogous.

(ii) Like before, we have

$$\|v_{E,F_g^-}\|^0 = \|u_{T,A_g^-}\|^0.$$

Let us apply Lemma 4 to $u = O^{j+1}$, $z = \omega_{n_j/4} \cdot \tau_{7n_j/32}$ and $z_\eta \in M_{n_j/2}$ defined for $\eta = (\eta^{j+1}, \dots, \eta^q) \in D_{j+1} \times \cdots \times D_q$ by

$$z_\eta = (u_{\eta^{j+1}})_{S,A} \qquad \text{if } \eta^{j+1} \neq g,$$

$$z_\eta = 0 \qquad \text{if } \eta^{j+1} = g.$$

(Here $u_{\eta^{j+1}}$ is defined by (29).)

By Lemma 5(ii) and, by the fact that $A \in \nabla_g^j$, we indeed have $z_\eta < z$ if $\eta^{j+1} \neq g$ (cf. (23) and (24)) and, if $\eta^{j+1} = g$, $z_\eta < z$ obviously. We see easily that the matrix y, obtained by (27) from the above u and z_η, is $\sim$ to u_{T,A_g^-}. Consequently, by Lemma 4,

$$u_{T,A_g^-} < O^{j+1} \omega_{n_j/4} \cdot \tau_{7n_j/32}.$$

By (7) and by Lemma 6,

$$O^{j+1} \cdot \omega_{n_j/4} \cdot \tau_{7n_j/32} \sim \omega_{n_{j+1} + \cdots + n_q} \cdot \omega_{n_j/4} \cdot \tau_{7n_j/32}$$

and this yields the first equality of (ii). The second one is completely analogous.

8. Finally we can conclude the proof of (28) and of the Proposition. We see that, by Lemma 8, (29) and by Lemma 9, if

$$
\begin{gathered}
g \in D_{j+1}, \quad \alpha, \beta \in \{-1,1\}^{N_j-1}, \quad e, f \in E_j, \quad A \in \nabla^j_j, \\
S = S_a \quad \text{for some } a \in \{-1,1\}^{n_j/4},
\end{gathered} \tag{30}
$$

then

$$
\begin{aligned}
\|\delta_{g,e,f,A,S,\alpha,\beta}\|_\wedge \leqq \zeta^{(1+\lambda)n_{j+1}} \|\omega_{n_{j+2}+\cdots+n_q}\|^0 \|\omega_{n_{j+2}+\cdots+n_q}\|^* \\
\cdot \|\vartheta_{n_{j+1}}\|^* \|\omega_{n_j/4}\|^0 \|\chi_{n_j/4}\|^* \|\tau_{7n_j/32}\|^0
\end{aligned} \tag{31}
$$

(we used the inequality $\|\omega_{n_{j+1}+n_{j+2}+\cdots+n_q}\|^0 \leqq \zeta^{(1+\lambda)n_{j+1}} \|\omega_{n_{j+2}+\cdots+n_q}\|^0$, which follows from (14)).

Clearly, $\beta_j - \gamma = 2^{-2N} \Sigma \delta_{g,e,f,A,S,\alpha,\beta}$, where the summation ranges over all tuples, satisfying (30). The number of such tuples is evidently equal to

$$
Q = |D_{j+1}| \cdot 2^{2N_j-1} \cdot |E_j|^2 \cdot 2^{n_j/4} \cdot 2^{n_j/4}.
$$

We obtain thus

$$
\|\beta_j - \gamma\|_\wedge \leqq 2^{-2N} \cdot Q \cdot \text{the right-hand side of (31)}.
$$

Clearly,

$$
\begin{gathered}
2^{2N} = 2^{2N_j-1} \cdot (2^{n_j})^2 \cdot (2^{n_{j+1}})^2 \cdot (2^{n_{j+2}+\cdots+n_q})^2, \\
|E_j|^2 = 2^{n_j} \quad \text{and} \quad |D_{j+1}| = 2^{n_{j+1}},
\end{gathered}
$$

and, by (9) and (13),

$$
\begin{gathered}
\|\omega_{n_{j+2}+\cdots+n_q}\|^0 \|\omega_{n_{j+2}+\cdots+n_q}\|^* = (2^{n_{j+2}+\cdots+n_q})^2, \\
\|\vartheta_{n_{j+1}}\|^0 \|\vartheta_{n_{j+1}}\|^* = 2^{n_{j+1}}, \\
\|\chi_{n_j/4}\|^0 \|\chi_{n_j/4}\|^* = 2^{n_j/2}.
\end{gathered}
$$

Making appropriate cancellations we arrive at

$$
\|\beta_j - \gamma\|_\wedge \leqq \zeta^{(1+\lambda)n_{j+1}} \|\omega^0_{n_j/4}\| \, \|\tau^0_{7n_j/32}\| \cdot \|\vartheta^0_{n_{j+1}}\|^{-1} \|\chi^0_{n_j/4}\|^{-1}. \tag{32}
$$

Since $\|\chi\|^0 = \|\tau^0\| \, \|\vartheta^0\|$, and by (14), $\|\omega_n\|^0 \leqq \zeta^{(1+\lambda)n}$ for every n, (32) and (9) yield

$$
\|\beta_j - \gamma\|_\wedge \leqq \zeta^{(1+\lambda)(n_{j+1}+n_j/4)} \cdot \|\tau^0\|^{7n_j/32} \|\vartheta^0\|^{-n_{j+1}-n_j/4} \leqq \zeta^{\lambda(n_{j+1}+n_j/4)-n_j/32}.
$$

By a completely analogous argument we get also

$$
\|\gamma - \beta_{j+1}\|_\wedge \leqq \zeta^{\lambda(n_{j+1}+n_j/4)-n_j/32}
$$

and, from these two inequalities, (28) follows.

We shall now define the sequence $\{n_j\}$. Let μ be such that

$$
1 < \mu < 1/32\lambda - 1/4 \quad \text{and} \quad \mu < 2 \tag{33}
$$

(notice that, by (14), $\lambda < 1/40$, thus $1/32\lambda - 1/4 > 1$). Finally, let $\gamma = \xi^{\lambda(\mu+1/4)-1/32}$.

By (33) we have $\gamma < 1$. Let $1 = p_1, p_2, \ldots$ be a fixed (i.e., independent of m) sequence of natural numbers such that

$$p_{i+1} \leqq \mu p_i \qquad \text{for every } i,$$
$$\sum \gamma^{p_i} < \infty.$$

Let us define $n_i = m \cdot p_i$. Evidently (19) and (20) are satisfied. Moreover, by (28),

$$\|\beta_j - \beta_{j+1}\| \leqq 2(\|\omega\|^p/\zeta)^{(\mu+1/4)n_j} \cdot \zeta^{-n_j/32} = 2\gamma^{n_j} = (2\gamma^{p_j})^m.$$

We see now that the assumptions of Lemma 2(ii) are satisfied: given $\eta > 0$ we choose $m = m(\eta)$ so that

$$2\sum_{j=1}^{\infty} (\gamma^{p_j})^m \leqq \eta \tag{34}$$

and take $k = 2^{2m}$. Next we choose q so that $2^{-N_q/2} \leqq \varepsilon$, where N_q is defined by (27). By (16), (17), (28) and (34), (5) is satisfied.

REMARK. A more careful computation yields that c_p for $p > 32$ (and, dually, for $p < 1 + 1/31$) do not have UAP. We would of course like to know whether the spaces c_p for all $p \neq 2$ do not have UAP.

It seems that to prove so, it would require a quite substantial change in our proof. For, looking at the inequality (32), we see that the only term which might disappear in an "ideal" situation (i.e., given the strongest possible lemma in line with Lemma 3) is the term $\|\tau_{2n,p/32}\|^p$, which comes from the inequality (23). Dropping this term in (32), we still get that λ should be $< \frac{1}{8}$. This, perhaps, indicates that for small p like $p = 4$, c_p might have UAP.

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